UNIVERSITY OF EXETER

YB-semitrusses and left non-degenerate solutions to the Yang-Baxter equation

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Overview

Main motivation

Find and describe left non-degenerate set-theoretic solutions of the Yang-Baxter equation.

The Yang-Baxter equation

- k - a field
- V - a k-vector space

A linear map $R : V \otimes V \to V \otimes V$ is a solution to the Yang-Baxter equation (YBE) if

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R).$$

Drienfeld, '92

``Maybe it would be interesting to study set-theoretical solutions to (9.1)''

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``Maybe it would be interesting to study set-theoretical solutions to (9.1)''

V.G. Drinfeld,
On some unsolved problems in quantum group theory,
Quantum Groups, Lecture Notes Math. 1510, Springer-Verlag, Berlin, 1992, 1–8.

Set-theoretic solutions

A set-theoretic solution (to the YBE) is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$

Write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

- r is left (resp. right) non-degenerate if λ_x (resp. ρ_x) is bijective, for any $x \in X$.
- non-degenerate if it is both left and right non-degenerate.

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Timeline of the study

- Involutive non-degenerate solutions
  - ['99] Etingof, Schedler & Soloviev
  - ['99] Gateva-Ivanova & Van den Bergh
    (Ring and group theoretical tools)
  - ['07] Rump
  - ['14] Cedó, Jespers & Okninski
    (Braces)
- Bijective non-degenerate solutions
  - ['00] Lu, Yan & Zhu
  - ['00] Soloviev
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  - ['17] Guarnieri & Vendramin
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Timeline of the study

- Not necessarily bijective solutions

['17] Lebed
(idempotent solutions)

['20] Cvetko-Vah & Verwimp
(cubic solutions and skew lattices)

['17] Catino, IC & Stefanelli
(left non-degenerate solutions and left cancellative semi-braces)

['19] Jespers & Van Antwerpen
(degenerate solutions and semi-braces)

['21] IC, Jespers, Van Antwerpen & Verwimp
(left non-degenerate solutions $\longleftrightarrow$ YB-semitruss)

Examples

Set-theoretic solutions

Example (Rump, 2007)

Let $(R, +, \cdot)$ be a radical ring

$$r(x, y) = (-x + x \circ y, \overline{(-x + x \circ y))} \circ x \circ y)$$

is an involutive non-degenerate solution.

Example (Rump, 2007)

Let $(R, +, \cdot)$ be a radical ring

Put $x \circ y = x + xy + y$.

R is radical if $(R, \circ)$ is a group

$$r(x, y) = (-x + x \circ y, \overline{(-x + x \circ y)} \circ x \circ y)$$

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Skew braces

A skew brace is a triple $(B, +, \circ)$ such that

- $(B, +)$ and $(B, \circ)$ be groups
- for any $a, b, c \in B$

$$a \circ (b + c) = a \circ b - a + a \circ c$$

If in addition $(B, +)$ is abelian $(B, +, \circ)$ is a brace or a skew brace of abelian type.

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Skew braces

Skew braces and solutions

Theorem (Guarnieri and Vendramin, 2017)

Let B be a skew brace. Define $r : B \times B \to B \times B$ by

$$r(x, y) = r(-x + x \circ y, \overline{(-x + x \circ y)} \circ x \circ y).$$

Then r is a bijective non-degenerate solution of the YBE. Moreover,

$$r \text{ is involutive} \iff (B, +) \text{ is abelian.}$$

The structure group

Definition (Etingof, Schedler and Soloviev, 1992

Let (X, r) be a solution. Define the structure group

$$G(X,r) = \mathrm{gr}(X \mid x \circ y = \lambda_x(y) \circ \rho_y(x))$$

From a solution to a skew brace

Theorem (Smoktunowicz and Vendramin, 2018)

If (X, r) is a bijective non-degenerate solution, then there exists a unique skew brace structure on $G(X, r)$ such that its associated solution $r_{G(X,r)}$ satisfies

$$r_{G(X,r)}(\iota \times \iota) = (\iota \times \iota)r$$

where $\iota : X \to G(X, r)$ is the canonical map.

The main issue of this correspondence is that ι is not an injective map in general.

Example. Let X be a set, $f, g \in \mathrm{Sym}(X)$ such that $fg = gf$. It is easy to see that

$$r \text{ is injective} \iff fg = \mathrm{id} \iff r \text{ is involutive}$$

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The structure monoid

Definition

Let (X, r) be a solution. Define the structure monoid

$$M(X, r) = \langle X \mid x \circ y = \lambda_x(y) \circ \rho_y(x) \rangle$$

Examples

Structure groups and structure monoids

Solutions and the structure monoid

Theorem (Gateva-Ivanova and Majid, 2008)

If (X, r) is a solution, then it is possible to extend the solution r to $M(X, r)$, i.e.

$$r_{M(X,r)_{|X \times X}} = r$$

and the map $\iota : X \to M(X, r)$ is injective.

Clearly, we cannot expect $M(X, r)$ has a skew brace structure. However, we can have something with the same flavor.

The derived monoid

Let (X, r) be a solution. Define the (left) derived monoid

$$A(X, r) = \langle X \mid x + \lambda_x(y) = \lambda_x(y) + \lambda_{\lambda_x(y)}\rho_y(x)\rangle$$

Ex. Let X be a left non-degenerate solution. We can define the (left) derived solution (X, s) in the following way

$$s : X \times X \to X \times X, \quad (x, y) \mapsto (y, \lambda_y\rho_{\lambda_x^{-1}(y)(x)}).$$

It is easy to see that $A(X, r) = M(X, s)$.

The connection between structure monoid and derived monoid

Theorem (Lu, Yan and Zhu 2000, Soloviev 2000, Jespers, Kubat and Van Antwerpen 2019, Cedó, Jespers and Verwimp 2021)

Let (X, r) be a left non-degenerate solution. Then λ induces a monoid homomorphism

$$\lambda : (M(X, r), \circ) \to \mathrm{Aut}(A(X, r), +)$$

with $\lambda_x(y) = \lambda_x(y)$ and there exists $\pi : M(X, r) \to A(X, r)$ a bijective 1-cocycle with respect to λ (i.e. $\pi(a \circ b) = \pi(a) + \lambda_a(\pi(b))$)

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Hence, we have an embedding

$$M(X,r) \to A(X,r) \rtimes \mathrm{Im}(\lambda).$$

Identifying $a \in M(X,r)$ with $\pi(a)$. The set $M(X,r)$ is equipped with two monoid operations

- $(M(X,r), \circ)$ where $\circ$ is the monoid operation of the structure monoid
- $(M(X,r), +)$ where $a + b = a \circ \lambda_a^{-1}(b)$

and satisfies

$$a \circ (b + c) = a \circ b + \lambda_a(c).$$

Definition (Brzeziński, 2018)

A semitruss is a quadruple $(A, +, \circ, \lambda)$ s.t.

- $(A, +)$ and $(A, \circ)$ are non-empty semigroups
- $\lambda : A \to \mathrm{Map}(A, A), a \mapsto \lambda_a$ is a mapping s.t.

$$\forall a, b, c \in A, \quad a \circ (b + c) = a \circ b + \lambda_a(c)$$

Ex. Let (X, r) be a left non-degenerate solution $(M(X, r), +, \circ, \lambda)$ is a semitruss.

It is easy to prove that $(M(X,r),+,\circ,\lambda)$ is a left semitruss such that

- $a + \lambda_a(b) = a \circ b$
- $\lambda : (M(X,r),\circ) \to \mathrm{Aut}(M(X,r),+)$, $a \mapsto \lambda_a$ is a semigroup morphism
- $\sigma : (M(X,r),+) \to \mathrm{End}(M(X,r),+)$, $a \mapsto \sigma_a$ is a semigroup anti-morphism such that $a + b = b + \sigma_a(b)$, where $\sigma_a(b) = \lambda_a \rho_{\lambda_b^{-1}(a)}(b)$.
- $\lambda_a \sigma_b = \sigma_{\lambda_a(b)} \lambda_b$

This leads to

Definition (IC, Jespers, Van Antwerpen, Verwimp, 2021)

A tuple $(A, +, \circ, \lambda, \sigma)$ is a YB-semitruss if

- $(A, +, \circ, \lambda)$ is a semitruss
- $\sigma : A \to \mathrm{Map}(A, A), a \mapsto \sigma_a$ is a mapping

s.t

- $\lambda_a \in \mathrm{Aut}(A, +)$ and $\lambda_a \lambda_b = \lambda_{a \circ b}$
- $a + \lambda_a(b) = a \circ b$
- $a + b = b + \sigma_b(a)$
- $\sigma_a \in \mathrm{End}(A, +)$ and $\sigma_{a+b} = \sigma_b \sigma_a$
- $\sigma_{\lambda_a(c)} \lambda_a(b) = \lambda_a \sigma_c(b)$

Examples

- Any skew brace $(B, +, \circ)$ is a YB-semitruss, with respect to the λ map and σ defined by $\sigma_a(b) = -b + a + b$.
- Let (X, r) be a left non-degenerate solution. Then $(M(X, r), +, \circ, \lambda, \sigma)$ is a YB-semitruss with $\sigma_b(a) = \lambda_b \rho_{\lambda_a^{-1}(b)}(a)$.

  Ex. For $r(x, y) = (y, y)$, we get $x \circ y = y \circ y$, $x + y = y + y$, $\lambda_x(y) = y$ and $\sigma_y(x) = \rho_y(x) = y$.
- Let A be a set, $\lambda : A \to \mathrm{Sym}(A)$ satisfying $\lambda_a \lambda_b = \lambda_{\lambda_a(b)}$. Define $a + b = b$ (note that$\lambda_a \in \mathrm{Aut}(A, +)$) and $a \circ b = \lambda_a(b)$, $\sigma_b(a) = b$. Then $(A, +, \circ, \lambda, \sigma)$ is aYB-semitruss.

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- Let A be a set, $\lambda : A \to \mathrm{Sym}(A)$ satisfying $\lambda_a \lambda_b = \lambda_{\lambda_a(b)}$. Define $a + b = b$ (note that$\lambda_a \in \mathrm{Aut}(A, +)$) and $a \circ b = \lambda_a(b)$, $\sigma_b(a) = b$. Then $(A, +, \circ, \lambda, \sigma)$ is aYB-semitruss.

The associated solution to a YB-semitruss

Theorem (IC, Jespers, Van Antwerpen, Verwimp, 2021)

Let $(A, +\circ, \lambda, \sigma)$ a YB-semitruss. Then

$$r(x,y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}\sigma_{\lambda)x(y)}(x))$$

is a left non-degenerate solution

Moreover,
Let (X, r) be a left non-degenerate solution then the structure monoid is a YB-semitruss and the solution associated to it is precisely the solution defined by Gateva-Ivanova and Majid with

$$r_{M(X,r)_{|X\times X}} = r$$

The associated solution to a YB-semitruss

When it is bijective?

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss.
r_A is bijective $\iff$ s_A is bijective $\iff$ $\forall a \in A$, σ_a is bijective

Theorem

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss with σ_a bijective ($\forall a \in A$). Then

$$r_A^{-1}(a, b) = (\sigma_a^{-1}\lambda_a(b), \lambda^{-1}_{\sigma_a^{-1}\lambda_a(b)}(a))$$

Moreover, $(A, +^{op}, \circ, \bar{\lambda}, \bar{\sigma})$ is a YB-semitruss with $\bar{\lambda}_a = \sigma_a^{-1}\lambda_a$ and $\bar{\sigma}_a = \sigma_a^{-1}$.
Its associated solution is r_A^{-1}.

A question

Let (X, r) be a left non-degenerate solution. What is the relation between (X, r) being right non-degenerate and r being bijective?

- Any finite involutive left non-degenerate solution is non-degenerate.

  [Rump, 2005]

  [Jespers and Okniński, 2005]

- An example of an infinite involutive solution that is left non-degenerate but not right non-degenerate.

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- Any non-degenerate solution such that $\lambda_x = \lambda_y$ implies $x = y$ is bijective.

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An application

Theorem

If (X, r) is a finite left non-degenerate solution, then

$$r \text{ is bijective} \iff (X, r) \text{ is right non-degenerate}$$

The implication $\Rightarrow$ has been proven by Castelli, Catino and Stefanelli.

The implication $\Leftarrow$ is based on the fact that the structure monoid is a YB-semitruss.

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